1. *Find the Lebesgue points of the function*

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0. \end{cases}$$

Solution: By the definition if x is a Lebesgue point of f if and only if

$$\lim_{r \to 0} \frac{1}{|E_r|} \int_{E_r} f(y)dy = f(x)$$

for any family of sets $\{E_r\}_{r>0}$ such that $E_r \subset B(x,r)$ and $|E_r|$ denotes the measure of the set E_r. Let $x > 0$. Now we know that the sets $B(x,r)$ lies inside $(0,\infty)$ whenever $r < x$. For any $r < x$, we have

$$\frac{1}{|E_r|} \int_{E_r} f(y)dy = \frac{1}{|E_r|} \int_{E_r} 1 \ dy = 1 = f(x).$$

Using a similar idea we get that for $x < 0$ and for all $r < -x$ we have

$$\frac{1}{|E_r|} \int_{E_r} f(y)dy = \frac{1}{|E_r|} \int_{E_r} 0 \ dy = 0 = f(x).$$

Now it is remaining to show that $x = 0$ is not a Lebesgue point. Consider the family $E_n = B(0, \frac{1}{n})$ for $n \in \mathbb{N}$. Now we can see that

$$\frac{1}{|E_n|} \int_{E_n} f(y)dy = \frac{1}{|E_n|} \Big(\int_{-\frac{1}{n}}^{0} f(y)dy + \int_{0}^{\frac{1}{n}} f(y)dy \Big) = \frac{1}{(2/n)}(0 + 1/n) = \frac{1}{2} \neq f(0).$$

□

2. *Consider the set $M = \{\sum_{n=-N}^{N} c_j e^{ijx} : c_1, c_2, ..., c_N \in \mathbb{C}\}$ where N is a given positive integer. Is this a closed subspace of $L^1(\mu)$, (where μ is the normalized Lebesgue measure in $[0, 2\pi]$)?. Justify.*

Solution: $M \subsetneq L^1(\mu)$ is not a closed subspace of $L^1(\mu)$. To prove this, for any $f \in L^1(\mu)$, we will find a sequence of functions in M which converges to f. Let F_N be the Fejer kernel given by $F_N(x) = \sum_{j=-N}^{N} \Big(1 - \frac{|j|}{N+1}\Big) e^{ijx}$. Clearly F_N belongs to M for all $N \in \mathbb{N}$. It is easy to see that $F_N(x) = \sum_{n\in\mathbb{Z}} \hat{F_N}(n) e^{inx}$ and for any $f \in L^1(\mu)$, $f * F_N(x) = \sum_{n\in\mathbb{Z}} \hat{F_N}(n) \hat{f}(n) e^{inx}$. It gives that $f * F_N$ also belongs to M. By a straight forward verification we can see that F_N is also an approximate identity in $L^1(\mu)$ and $f * F_N$ converges to f in $L^1(\mu)$ as $N \to \infty$.

□

3. *Prove that*

$$x = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{cos(2n-1)x}{(2n-1)^2}$$

for $0 \leq x \leq \pi$.

Solution: Let $f(x) = x$ with $0 \leq x \leq \pi$. Let f_1 and f_2 be its odd and even periodic extensions on $[-\pi, \pi]$. We know that $f_1(x) = x$ and $f_2(x) = |x|$. The Fourier transform of f_2 can be calculated as $\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{cos(2n-1)x}{(2n-1)^2}$. Since $f_2(x) = x = f(x)$ on $[0, \pi]$ and $f \in C^2([0, \pi])$, we have $x = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{cos(2n-1)x}{(2n-1)^2}$. □

4. *Let* $f \in L^1(\mu)$ *and* $S_N(x) = \sum_{n=-N}^{N} \hat{f}(n)e^{inx}$. *Show that* $lim_{N\to\infty} \frac{S_N(x)}{N}$ *exists for every* x *and find the limit.*

Solution: By Riemann-Lebesgue lemma $\hat{f}(n) \to 0$ as $n \to \pm\infty$ for $f \in L^1(\mu)$. Now

$$\Big|\frac{S_N(x)}{N}\Big| \leq \sum_{n=-N}^{N} \Big|\frac{\hat{f}(n)}{N}\Big|.$$

Now we have the sequences $\{\hat{f}(n)\}_{n\in\mathbb{N}}$ and $\{\hat{f}(-n)\}_{n\in\mathbb{N}\cup\{0\}}$ converges absolutely to 0 and its average also converges to 0. It means that the sequence $\{\frac{S_N(x)}{2N+1}\}_N$ converges. Multiply and divide by $(2N+1)$ to $\frac{S_N(x)}{N}$ gives that the $\lim_{N\to\infty} \frac{S_N(x)}{N} = 0$.

□

5. *If* $f \in L^1(\mu)$ *(*μ *as in Problem 4) and if* f *is continuous at* 0 *show that* $\sum_{-N}^{N}(1 - \frac{|n|}{N+1})\hat{f}(n) \to f(0)$ *as* $N \to \infty$.

Solution: From the solution of 2 we can write

$$f * F_N(x) = \sum_{-N}^{N}(1 - \frac{|n|}{N+1})\hat{f}(n)e^{inx}$$

and $f * F_N$ converges to f in $L^1(\mu)$. Now

$$\begin{aligned} f * F_N(x) &= \int_{-\pi}^{\pi} f(x-y)F_N(y)dy \\ &= \int_{0}^{\pi} f(x-y)F_N(y)dy + \int_{0}^{\pi} f(x+y)F_N(y)dy \end{aligned}$$

Now see that the first integral converges to $\frac{1}{2}f(x-)$ and second integral converges to $\frac{1}{2}f(x+)$. So if f is continuous at x then $f * F_N(x)$ goes to $f(x)$. Now it is easy to see with the case $x = 0$.

□

6. *If* $\sum_{n=1}^{\infty} |a_n cos(nx) + b_n sin(nx)| < \infty$ *for all* x *in a set of positive measure show that* $\sum_{n=1}^{\infty} |a_n| < \infty$ *and* $\sum_{n=1}^{\infty} |b_n| < \infty$.

Solution: Proceed as in Cantor-Lebesgue Theorem and use the fact that $\int_E |cos(nx - \theta_n)|dx \geq \int_E cos^2(nx - \theta_n)$.